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THE SHIFT ON THE INVERSE LIMIT OF A COVERING PROJECTION†

BY

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ABSTRACT

A group endomorphism $\alpha : G \to G$ is said to be *weakly shift equivalent* to the group endomorphism $\beta : H \to H$ if there exists $h \in H$ such that α is shift equivalent to $\mathrm{Ad}[h] \circ \beta$. Given covering projections $a : X \to X$, $b : Y \to Y$ of compact, connected, locally path connected, semilocally simply connected metric spaces with fixed points $x_0 \in X$, $y_0 \in Y$ respectively, the inverse limits

$$\Sigma_a = \varprojlim (X, a) = \{(x_i)_{i \in Z^+}, ax_{i+1} = x_i, i \in Z^+\},$$

$$\Sigma_b = \varprojlim (Y, b) = \{(y_i)_{i \in Z^+}, by_{i+1} = y_i, i \in Z^+\}$$

and the "shift" maps $\sigma_a : \Sigma_a \to \Sigma_a$, $\sigma_b : \Sigma_b \to \Sigma_b$ defined by $\sigma_a((x_i)_{i \in Z^+}) = (x_{i+1})_{i \in Z^+} \in \Sigma_a$, $\sigma_b((y_i)_{i \in Z^+}) = (y_{i+1})_{i \in Z^+} \in \Sigma_b$ are considered. It is proven that if σ_a and σ_b are topologically conjugate then $a_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is weakly shift equivalent to $b_* : \pi_1(Y, y_0) \to \pi_1(Y, y_0)$. Furthermore, if $a : X \to X$ and $b : Y \to Y$ are expanding endomorphisms of compact differentiable manifolds, weak shift equivalence is a complete invariant of topological conjugacy. The use of this invariant is demonstrated by giving a complete classification of the shifts of expanding maps on the klein bottle. The reader is referred to Section 4 of this work for a detailed statement of results.

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1. Generalities on shift equivalence and weak shift equivalence

Given group endomorphims $\alpha : G \to G$ and $\beta : H \to H$, α is said to be *shift equivalent* to β iff there exist group homomorphisms $\phi : G \to H$, $\psi : H \to G$ and $n \in Z^+$ such that

$$\phi\alpha = \beta\phi, \quad \psi\beta = \alpha\psi, \quad \psi\phi = \alpha^n, \quad \phi\psi = \beta^n$$

([6], [7]).

1.1. Remark. It should be noticed that if ϕ is an injective map then the conditions $\phi\alpha = \beta\phi$, $\phi\psi = \beta^n$ together imply the rest.

1.2. Definition. Given group endomorphisms $\alpha : G \to G$ and $\beta : H \to H$, α is said to be *weakly shift equivalent* to β if there exists $h \in H$ such that α is shift equivalent to $\mathrm{Ad}[h] \circ \beta$.

1.3. Proposition. *Shift equivalence is an equivalence relation.*

Proof. Symmetry and reflexivity are obvious. We prove that shift equivalence is a transitive relation: Assume that $\alpha : G \to G$ is shift equivalent to $\beta : H \to H$ and $\beta : H \to H$ is shift equivalent to $\gamma : K \to K$. There exist group endomorphisms $\phi : G \to H$, $\psi : H \to G$, $\phi' : H \to K$, $\psi' : K \to H$, m, $n \in Z^+$ such that

$$\phi\alpha = \beta\phi, \quad \psi\beta = \alpha\psi, \quad \psi\phi = \alpha^m, \quad \phi\psi = \beta^m,$$

$$\phi'\beta = \gamma\phi', \quad \psi'\gamma = \beta\psi', \quad \psi'\phi' = \beta^n, \quad \phi'\psi' = \gamma^n.$$

Writing

$$\phi'' = \phi'\phi \quad \text{and} \quad \psi'' = \alpha^{2mn-m-n}\psi\psi'$$

we obtain

$$\phi''\alpha = \gamma\phi'', \quad \psi''\gamma = \alpha\psi'', \quad \psi''\phi'' = \alpha^{2mn}, \quad \phi''\psi'' = \gamma^{2mn}.$$

1.4. Proposition. *Weak shift equivalence is an equivalence relation among injective group endomorphisms.*

Proof. In view of 1.3 it is sufficient to show that if $\mathrm{Ad}[g] \circ \alpha$ is shift equivalent to $\mathrm{Ad}[h] \circ \beta$ then α is shift equivalent to $\mathrm{Ad}[\phi(g^{-1})h] \circ \beta$: Let $\mathrm{Ad}[g] \circ \alpha$ be shift equivalent to $\mathrm{Ad}[h] \circ \beta$. There exist $\phi : G \to H$, $\psi : H \to G$ and $n \in Z^+$ such that

$$\phi \circ (\mathrm{Ad}[g] \circ \alpha) = (\mathrm{Ad}[h] \circ \beta) \circ \phi, \tag{1}$$

$$\psi \circ (\mathrm{Ad}[h] \circ \beta) = (\mathrm{Ad}[g] \circ \alpha) \circ \psi, \tag{2}$$

$$\psi \circ \phi = (\mathrm{Ad}[g] \circ \alpha)^n, \tag{3}$$

$$\phi \circ \psi = (\mathrm{Ad}[h] \circ \beta)^n. \tag{4}$$

We notice that as α, β are injective, ϕ, ψ are injective, too. (1) can be regrouped to give

$$\phi \circ \alpha = (\mathrm{Ad}[\phi(g^{-1})h] \circ \beta) \circ \phi. \tag{5}$$

Define $h_1^{(i)}, h_2^{(i)} \in H$ for $i \in Z^+$ inductively by

$$h_1^{(0)} = h_2^{(0)} = \text{the neutral element in } H,$$

$$h_1^{(i+1)} = h\beta(h_1^{(i)}),$$

$$h_2^{(i+1)} = \phi(g^{-1})h\beta(h_2^{(i)}).$$

Consequently

$$(\mathrm{Ad}[h] \circ \beta)^i = \mathrm{Ad}[h_1^{(i)}] \circ \beta^i,$$

$$(\mathrm{Ad}[\phi(g^{-1})h] \circ \beta)^i = \mathrm{Ad}[h_2^{(i)}] \circ \beta^i,$$

for any $i \in Z^+$. We claim

$$h_2^{(i)}(h_1^{(i)})^{-1} \in \phi(G) \quad \text{for each } i \in Z^+$$

which can be inductively proven as follows: Clearly

$$h_2^{(0)}(h_1^{(0)})^{-1} \in \phi(G).$$

Assuming

$$h_2^{(i)}(h_1^{(i)})^{-1} \in \phi(G)$$

we find

$$\begin{aligned} h_2^{i+1}(h_1^{(i+1)})^{-1} &= \phi(g^{-1})h\beta(h_2^{(i)})(\beta(h_1^{(i)}))^{-1}h^{-1} \\ &= \phi(g^{-1})(\mathrm{Ad}[h] \circ \beta(h_2^{(i)}(h_1^{(i)})^{-1})) \\ &= \phi(g^{-1})(\mathrm{Ad}[h] \circ \beta \circ \phi(g')) \quad \text{for some } g' \in G \\ &= \phi(g^{-1})\phi(\mathrm{Ad}[g]\alpha(g')) \in \phi(G). \end{aligned}$$

Therefore, there exists $g'' \in G$ such that

$$\begin{aligned}(\mathrm{Ad}[\phi(g^{-1})h] \circ \beta)^n &= \mathrm{Ad}[h_2^{(n)}(h_1^{(n)})^{-1}] \circ (\mathrm{Ad}[h] \circ \beta)^n \\ &= \mathrm{Ad}[\phi(g'')] \circ \phi \circ \psi \\ &= \phi \circ (\mathrm{Ad}[g''] \circ \psi).\end{aligned}$$

Writing $\psi' = \mathrm{Ad}[g''] \circ \psi$ we obtain

$$\phi \circ \psi' = (\mathrm{Ad}[\phi(g^{-1})h] \circ \beta)^n. \tag{6}$$

By 1.1, (5) and (6) together imply that ϕ, ψ' effect a shift equivalence of α to $\mathrm{Ad}[\phi(g^{-1})h] \circ \beta$.

To my knowledge, this proposition is not known for endomorphisms which are not necessarily injective.

2. Generalities on inverse limits of covering projections

This part consists of statement of some results from [5] and a corollary thereof which will be used in the sequel.

By an *inverse sequence* (X_i, a_{ij}) we shall understand a system consisting of morphisms $a_{ij}: X_i \to X_j$, $i, j \in Z^+$, $i \geqq j$ with the property that $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ for any $i, j, k \in Z^+$ with $i \geqq j \geqq k$ and $\alpha_{ii} = I$ the identity morphism on X_i, for any $i \in Z^+$.

If $\mathscr{A} = (X_i, a_{ij})$ is an inverse sequence of continuous maps of topological spaces, the *inverse limit* $\Sigma_a = \varprojlim (X_i, a_{ij})$ of $\mathscr{A}$ is the subspace of $\Pi(X_i, i \in Z^+)$ consisting of those sequences $(x_i)_{i \in Z^+}$ with $a_{ij}(x_i) = x_j$ for any $i, j \in Z^+$ with $i \geqq j$. Σ_a has a natural topology as a subspace of $\Pi(X_i, i \in Z^+)$. Sets of the form

$$[V]^{(n)} = \{(x_i)_{i \in Z^+} \in \Sigma_a, x_n \in V\}$$

where $n \in Z^+$ and V is a neighbourhood of $x_n \in X_n$ constitute a base for the topology of Σ_a.

In a topological space X a neighbourhood V of $x \in X$ will be said to be a *good neighbourhood* of x provided that V is path connected and

$$i_* : \pi_1(V, x) \to \pi_1(X, x)$$

induced by the inclusion

$$i : V \to X$$

is trivial.

Let (X_i, a_{ij}), (Y_i, b_{ij}), (Z_i, c_{ij}) be inverse sequences of covering projections of compact, connected, locally path connected, semilocally simply connected metric spaces and

$$\Sigma_a = \varprojlim (X_i, a_{ij}), \quad \Sigma_b = \varprojlim (Y_i, b_{ij}), \quad \Sigma_c = \varprojlim (Z_i, c_{ij}),$$

$$\xi = (x_i)_{i \in Z^+} \in \Sigma_a, \quad \eta = (y_i)_{i \in Z^+} \in \Sigma_b, \quad \zeta = (z_i)_{i \in Z^+} \in \Sigma_c.$$

Given a continuous map $f: \Sigma_a \to \Sigma_b$ with $f(\xi) = \eta$, $n \in Z^+$ and a good neighbourhood V of $y_0 \in Y_0$, consider the projection $q_n: \Sigma_b \to Y_n$ defined by $q_n((y_i')_{i \in Z^+}) = y_n'$ for any $(y_i')_{i \in Z^+} \in \Sigma_b$ and the path component V_n of $(b_{n0})^{-1}V$ containing $y_n \in Y_n$. It can be verified that b_{n0} restricted to V_n is a homeomorphism onto V. Under these conditions there exists $m \in Z^+$ and a neighbourhood U of x_m such that

$$q_n \circ f([U]^{(m)}) \subseteq V_n.$$

Consequently if $\langle \lambda \rangle \in \pi_1(X_m, x_m)$ and $\hat{\lambda} = (\lambda_i)_{i \in Z^+}$ is the unique path in Σ_a with $\hat{\lambda}(0) = \xi$ and $\lambda_m = \lambda$, then $q_n \circ f(\hat{\lambda})(1) \in V_n$. We define

$$L(m, n, V) f(\langle \lambda \rangle) = \langle q_n \circ f(\hat{\lambda}) * \omega \rangle \in \pi_1(Y_n, y_n)$$

where ω is any path in V_n with $\omega(0) = q_n \circ f(\hat{\lambda}(1))$ and $\omega(1) = y_n$.

$$L(m, n, V) f: \pi_1(X_m, x_m) \to \pi_1(Y_n, y_n)$$

is a well-defined map; in particular it is independent of the choice of ω. The fundamental properties of $L(m, n, V)f$ can be summarised as follows:

2.1. LEMMA. (1.1 in [5]) *Let $f: \Sigma_a \to \Sigma_b$ be a continuous function with $f(\xi) = \eta$. For any $n \in Z^+$ and any good neighbourhood V of $y_0 \in Y_0$, there exists a map*

$$L(m, n, V) f: \pi_1(X_m, x_m) \to \pi_1(Y_n, y_n)$$

for some $m \in Z^+$ such that

(a) *if $L(m, n, V)f$ is defined and $m' \geqq m$, then*

$$L(m', n, V) f: \pi_1(X_{m'}, x_{m'}) \to \pi_1(Y_n, y_n)$$

is defined and

$$L(m', n, V)f = L(m, n, V)f \circ (a_{m'm})_\# ;$$

(b) *if* $L(m, n, V)f$ *is defined and* $n' \leqq n$ *then*

$$L(m, n', V)f\colon \pi_1(X_m, x_m) \to \pi_1(Y_{n'}, y_{n'})$$

is defined and

$$L(m, n', V)f = (b_{nn'})_\# \circ L(m, n, V)f;$$

(c) $L(m, n, V)f$ *is a group homomorphism for all sufficiently large* $m \in Z^+$.

2.2. Lemma. (1.2 in [5])

(a) *If* U *is any good neighbourhood of* $x_0 \in X_0$ *and* $I : \Sigma_a \to \Sigma_a$ *is the identity map, then*

$$L(m, n, U)I : \pi_1(X_m, x_m) \to \pi_1(X_n, x_n)$$

is defined for all $m, n \in Z^+$ *with* $m \geqq n$ *and satisfies*

$$L(m, n, U)I = (a_{mn})_\#;$$

(b) *If* $f: \Sigma_a \to \Sigma_b$ *and* $g : \Sigma_b \to \Sigma_c$ *are continuous maps with* $f(\xi) = \eta$ *and* $g(\eta) = \zeta$ *and* V, W *are good neighbourhoods of* y_0, z_0 *respectively and given* n, $p \in Z^+$ *such that*

$$L(n, p, W)g : \pi_1(Y_n, y_n) \to \pi_1(Z_p, z_p)$$

is defined, then for all sufficiently large $m \in Z^+$

$$L(m, n, V)f\colon \pi_1(X_m, x_m) \to \pi_1(Y_n, y_n)$$

and

$$L(m, p, W)(g \circ f) : \pi_1(X_m, x_m) \to \pi_1(Z_p, z_p)$$

are defined and satisfy

$$L(m, p, W)(g \circ f) = L(n, p, W)g \circ L(m, n, V)f.$$

2.3. Corollary. *Let* $f\colon \Sigma_a \to \Sigma_b$ *be a homeomorphism with* $f(\xi) = \eta$. *For any* $n \in Z^+$ *and any good neighbourhood* V *of* y_0, $L(m, n, V)f$ *is defined and injective for all sufficiently large* $m \in Z^+$.

Proof. Given $n \in Z^+$, choose $M \in Z^+$ such that $L(M, n, V)f$ is defined.

Let U be any good neighbourhood of $x_0 \in X_0$. There exists $N \in Z^+$ such that $L(N, M, U)f^{-1}$ is defined and

$$L(M, n, V)f \circ L(N, M, U)f^{-1} = L(N, n, V)(f \circ f^{-1})$$
$$= (b_{Nn})_*$$

by 2.2. Similarly there exists $m \in Z^+$ such that $L(m, N, V)f$ is defined and

$$L(N, M, U)f^{-1} \circ L(m, N, V)f = L(m, M, U)(f^{-1} \circ f)$$
$$= (a_{mM})_*.$$

We claim that $L(m, n, V)f$ is injective: Assume

$$L(m, n, V)f(\langle\lambda_1\rangle) = L(m, n, V)f(\langle\lambda_2\rangle).$$

Therefore

$$(b_{Nn})_* \circ L(m, N, V)f(\langle\lambda_1\rangle) = (b_{Nn})_* \circ L(m, N, V)f(\langle\lambda_2\rangle)$$

by 2.1(b). Consequently

$$L(m, N, V)f(\langle\lambda_1\rangle) = L(m, N, V)f(\langle\lambda_2\rangle)$$

as $(b_{nN})_*$ is an injective map. Applying $L(N, M, U)f^{-1}$ on both sides

$$L(N, M, U)f^{-1} \circ L(m, N, V)f(\langle\lambda_1\rangle) = L(N, M, U)f^{-1} \circ L(m, N, V)f(\langle\lambda_2\rangle)$$

hence

$$(a_{mM})_*(\langle\lambda_1\rangle) = (a_{mM})_*(\langle\lambda_2\rangle)$$

from which we conclude

$$\langle\lambda_1\rangle = \langle\lambda_2\rangle$$

as $(a_{mM})_*$ is an injective map. Thus $L(m, n, V)f$ is an injective map. For any $m' \geqq m$

$$L(m', n, V)f = L(m, n, V)f \circ (a_{m'm})_*$$

is injective.

3. Inverse limit of a single covering projection and the "shift"

Let X, Y be compact, connected, locally path connected, semilocally simply connected metric spaces, $a : X \to X$, $b : Y \to Y$ be covering projections with fixed points $x_0 \in X$, $y_0 \in Y$ respectively. In the following we shall consider inverse sequences (X_i, a_{ij}), (Y_i, b_{ij}) where $X_i = X$, $Y_i = Y$, $a_{ij} = a^{i-j}$, $b_{ij} = b^{i-j}$ for all $i, j \in Z^+$ with $i \geqq j$. Thus

$$\Sigma_a = \{(x_i)_{i \in Z^+} \in X^{Z^+}, a(x_{i+1}) = x_i, i \in Z^+\},$$

$$\Sigma_b = \{(y_i)_{i \in Z^+} \in Y^{Z^+}, b(y_{i+1}) = y_i, i \in Z^+\}.$$

Further, we write $\xi_0 = (x_0)_{i \in Z^+}$, $\eta_0 = (y_0)_{i \in Z^+}$. Such simple inverse limits are of importance mainly owing to some homeomorphisms naturally attached to them. The map

$$\hat{a} : \Sigma_a \to \Sigma_a$$

defined by

$$\hat{a}((x_i)_{i \in Z^+}) = (ax_i)_{i \in Z^+}$$

is a homeomorphism the inverse whereof

$$\sigma_a((x_i)_{i \in Z^+}) = (x_{i+1})_{i \in Z^+}$$

has come to be called the "shift" on Σ_a. We note that a point $\xi' = (x'_i)_{i \in Z^+} \in \Sigma_a$ is a fixed point of $\hat{a}$ iff $x'_i = x'_0$ for all $i \in Z^+$ and x'_0 is a fixed point of a.

Similarly we define $\hat{b}$, $\sigma_b = \hat{b}^{-1}$ and note that $\sigma_b(\eta_0) = \eta_0$.

4. Statement of results

The purpose of this paper is to prove the following results:

4.1. THEOREM. *If there exists a homeomorphism* $f : \Sigma_a \to \Sigma_b$ *such that* $f(\xi_0) = \eta_0$ *and* $f \circ \sigma_a = \sigma_b \circ f$ *then*

$$a_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is shift equivalent to

$$b_* : \pi_1(Y, y_0) \to \pi_1(Y, y_0).$$

4.2. THEOREM. *If* σ_a *is topologically conjugate to* σ_b *(that is, if there exists a homeomorphism* $f : \Sigma_a \to \Sigma_b$ *such that* $f \circ \sigma_a = \sigma_b \circ f$*) then*

$$a_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is weakly shift equivalent to

$$b_* : \pi_1(Y, y_0) \to \pi_1(Y, y_0).$$

4.3. THEOREM. *Let $\phi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $\psi : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ effect a shift equivalence of $a_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ to $b_* : \pi_1(Y, y_0) \to \pi_1(Y, y_0)$. If there exists a covering projection $f : X \to Y$ such that $f \circ a = b \circ f$, $f(x_0) = y_0$, $f_* = \phi$, then there exists a homeomorphism*

$$F : \Sigma_a \to \Sigma_b$$

such that

$$F \circ \sigma_a = \sigma_b \circ F$$

and

$$F(\xi_0) = \eta_0.$$

4.4. DEFINITION. ([4]) $f : X \to X$ is said to be an *expanding map* if X is a differentiable manifold on which there exists a riemannian metric $\langle \cdot, \cdot \rangle$ such that f is differentiable and there exist constants $C > 0$, $\lambda > 1$ such that

$$\| Tf^n(x)(v) \|_{f^n(x)} \geqq C\lambda^n \| v \|_x$$

for any $x \in X$, $v \in T_x X$, $n \in Z^+$, where $\| v \|_x$ stands for $\langle v, v \rangle_x$.

Notice that on a compact manifold being expanding is independent of the riemannian metric. In particular, if f is an expanding map on a compact manifold X and $h : X \to X$ is a diffeomorphim, then $h^{-1} \circ f \circ h$ is expanding. Expanding maps are covering projections and always possess fixed points. If the expanding map is a diffeomorphism, then the fixed point is unique ([4], Theorem 1, Lemma 3 in I).

4.5. THEOREM. *Let $a : X \to X$, $b : Y \to Y$ be expanding maps, $\xi_0 = (x_0)_{i \in Z^+} \in \Sigma_a$, $\eta_0 = (y_0)_{i \in Z^+} \in \Sigma_b$ be fixed points of σ_a, σ_b respectively. $a_* = \pi_1(X, x_0) \to \pi_1(X, x_0)$ is shift equivalent to $b_* : \pi_1(Y, y_0) \to \pi_1(Y, y_0)$ iff there exists a homeomorphism*

$$F : \Sigma_a \to \Sigma_b$$

such that $F \circ \sigma_a = \sigma_b \circ F$ and $F(\xi_0) = \eta_0$.

4.6. THEOREM. *Let* $a : X \to X$, $b : Y \to Y$ *be expanding maps*, $\xi_0 = (x_0)_{i\in Z^+} \in \Sigma_a$, $\eta_0 = (y_0)_{i \in Z^+} \in \Sigma_b$ *be fixed points of* σ_a, σ_b *respectively*. σ_a *is topologically conjugate to* σ_b *iff*

$$a_{\#} : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is weakly shift equivalent to

$$b_{\#} : \pi_1(Y, y_0) \to \pi_1(Y, y_0).$$

Theorems 4.1, 4.3 indicate that shift equivalence can be employed effectively in a category of topological spaces where covering projections abound in the sense that to any reasonable injective homomorphism

$$\phi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

a covering projection $f: X \to Y$ can be attached with $f(x_0) = y_0$ and $f_{\#} = \phi$. As an example of such a situation the nilmanifold endomorphisms [2] may be mentioned. On the other hand, efficiency of weak shift equivalence necessitates also an abundance of fixed points in the sense that if x_0 is fixed point of $f: X \to X$ and $\langle \lambda \rangle \in \pi_1(X, x_0)$, then there exists a fixed point x_0' of f such that

$$\mathrm{Ad}[\langle \lambda \rangle] \circ f_{\#} : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is shift equivalent to

$$f_{\#} : \pi_1(X, x_0') \to \pi_1(X, x_0').$$

Expanding maps will be shown to have the fixed point abundance.

It seems reasonable to expect that covering projection and fixed point abundance are correctly exhibited by infranilmanifold endomorphisms and anosov endomorphisms, the latter being possibly a subcategory of the former ([1], [3]).

The concept of shift equivalence can be defined in any category. It was introduced by R. F. Williams in his analysis of subshifts of the finite type and the so-called "expanding attractors" ([6], [7], [8]). Although many ideas with profound implications have been put forward, the situation as regards the above-mentioned problems appears to be far from clear.

5. Proofs

5.1. Lemma. *For any $m, n \in Z^+$ with $m \geqq n$ and any good neighbourhood U of x_0, $L(m, n, U)\hat{a}^k$ is defined for any $k \in Z$ with $m - n + k \geqq 0$ and*

$$L(m, n, U)\hat{a}^k = (a_*)^{m-n+k}.$$

Proof. Given $\langle \lambda \rangle \in \pi_1(X_m, x_m) = \pi_1(X, x_0)$ let $\hat{\lambda} = (\lambda_i)_{i \in Z^+}$ be the unique path in Σ_a with the property $\hat{\lambda}(0) = \xi_0$, $\lambda_m = \lambda$. If $p_n : \Sigma_a \to X_n = X$ is the projection defined by $p_n((x'_i)_{i \in Z^+}) = x'_n$ for any $(x'_i)_{i \in Z^+} \in \Sigma_a$, then

$$p_n \hat{a}^k \hat{\lambda} = a^{m-n+k} \lambda.$$

Consequently $L(m, n, U)\hat{a}^k$ is defined and equals $(a_*)^{m-n+k}$.

5.2. Lemma. *Let $f : \Sigma_a \to \Sigma_b$ be any continuous map with $f(\xi_0) = \eta_0$. For any $n \in Z^+$ and any good neighbourhood V of y_0, $L(m, n, V)(f \circ \hat{a})$ and $L(m, n, V)f$ are defined and satisfy*

$$L(m, n, V)(f \circ \hat{a}) = L(m, n, V)f \circ a_*$$

for all sufficiently large $m \in Z^+$.

Proof. Let $L(m', n, V)f$ be defined. By 2.2, for any good neighbourhood U of x_0 there exists $m \in Z^+$ such that $L(m, m', U)\hat{a}$ and $L(m, n, V)(f \circ \hat{a})$ are defined and satisfy

$$\begin{aligned} L(m, n, V)(f \circ \hat{a}) &= L(m', n, V)f \circ L(m, m', U)\hat{a} \\ &= L(m', n, V)f \circ (a_*)^{m-m'+1} \quad \text{by 5.1} \\ &= L(m, n, V)f \circ a_* \quad \text{by 2.1 (a).} \end{aligned}$$

5.3. Lemma. *Let $f : \Sigma_a \to \Sigma_b$ be any continuous function with $f(\xi_0) = \eta_0$. For any $n \in Z^+$ and any good neighbourhood V of y_0, $L(m, n, V)(\hat{b} \circ f)$ and $L(m, n, V)f$ are defined and satisfy*

$$L(m, n, V)(\hat{b} \circ f) = b_* \circ L(m, n, V)f$$

for all sufficiently large $m \in Z^+$.

Proof. $L(n, n, V)\hat{b} = b_*$ by 5.1. On the other hand, by 2.2 there exists $m \in Z^+$ such that $L(m, n, V)f$, $L(m, n, V)(\hat{b} \circ f)$ are defined and satisfy

$$L(m,n,V)(\hat{b}\circ f)=L(n,n,V)\hat{b}\circ L(m,n,V)f$$
$$=b_*\circ L(m,n,V)f.$$

5.4. Corollary. *If* $f:\Sigma_a\to\Sigma_b$ *is a continuous map with* $f(\xi_0)=\eta_0$ *and* $f\circ\hat{a}=\hat{b}\circ f$ *then for any* $n\in Z^+$ *and any good neighbourhood* V *of* $y_0\in Y$, $L(m,n,V)f$ *is defined and satisfies*

$$L(m,n,V)f\circ a_*=b_*\circ L(m,n,V)f$$

for all sufficiently large $m\in Z^+$.

Proof. This is a direct consequence of 5.2 and 5.3.

5.5. Proof of 4.1. Given $p\in Z^+$ and any good neighbourhood U of $x_0\in X$, by 2.1 and 5.4 we can choose $n\in Z^+$ such that

$$\psi=L(n,p,U)f^{-1}:\pi_1(Y,y_0)\to\pi_1(X,x_0)$$

is defined as an injective group homomorphism such that

$$\psi\circ b_*=a_*\circ\psi.$$

Let V be any good neighbourhood of $y_0\in Y$. By 2.2(b) there exists $m\in Z^+$ such that $\phi=L(m,n,V)f$ is defined as a group homomorphism and satisfies

$$\psi\circ\phi=L(n,p,U)f^{-1}\circ L(m,n,V)f$$
$$=L(m,p,U)(f^{-1}\circ f)$$
$$=(a_*)^{m-p}.$$

In view of 1.1 we conclude that ϕ, ψ effect a shift equivalence of a_* and b_*.

5.6. Lemma. *Let* y_0, y_0' *be fixed points of a continuous map* $g:Y\to Y$. *There exists* $\langle\mu\rangle\in\pi_1(Y,y_0)$ *and an isomorphism* $\gamma:\pi_1(Y,y_0)\to\pi_1(Y,y_0')$ *such that*

$$\gamma\circ\mathrm{Ad}[\langle\mu\rangle]\circ g_*=g_*\circ\gamma.$$

Proof. Let τ be a path in Y with the property $\tau(0)=y_0'$, $\tau(1)=y_0$. Define

$$\gamma:\pi_1(Y,y_0)\to\pi_1(Y,y_0')$$

by

$$\gamma(\langle\lambda\rangle)=\langle\tau*\lambda*\tau^{-1}\rangle$$

for any $\langle\lambda\rangle \in \pi_1(Y, y_0)$. γ is obviously an isomorphism. On the other hand

$$\gamma^{-1} \circ g_* \circ \gamma(\langle\lambda\rangle) = \langle\tau^{-1} * (g\tau) * (g\lambda) * (g\tau^{-1}) * \tau\rangle.$$

Writing $\mu = \tau^{-1} * g\tau$ we find that

$$\gamma^{-1} \circ g_* \circ \gamma = \mathrm{Ad}[\langle\mu\rangle] \circ g_*.$$

5.7. Proof of 4.2. Let $F : \Sigma_a \to \Sigma_b$ be a homeomorphism with $F \circ \hat{a} = \hat{b} \circ F$. There exists a fixed point $y_0' \in Y$ of b such that $\eta_0' = (y_0')_{i \in Z^+}$ is a fixed point of $\hat{b}$ and $F(\xi_0) = \eta_0'$. By 4.1, $a_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is shift equivalent to $b_* : \pi_1(Y, y_0') \to \pi_1(Y, y_0')$. Combining with 5.6 we conclude that $a_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is weakly shift equivalent to $b_* : \pi_1(Y, y_0) \to \pi_1(Y, y_0)$.

5.8. Proof of 4.3. Assume $\phi \circ a_* = b_* \circ \phi$, $\psi \circ b_* = a_* \circ \psi$, $\psi \circ \phi = a_*^n$, $\phi \circ \psi = b_*^n$. Let $g : Y \to X$ be the unique map satisfying $f \circ g = b^n$ and $g(y_0) = x_0$. From $f \circ g = b^n$ and $f \circ a = b \circ f$ we obtain $f \circ g \circ f = b^n \circ f = f \circ a^n$ and

$$f \circ g \circ b = b^{n+1} = b \circ f \circ g = f \circ a \circ g.$$

As f is a covering projection, these imply

$$g \circ f = a^n \quad \text{and} \quad g \circ b = a \circ g$$

respectively. Finally define

$$F : \Sigma_a \to \Sigma_b \quad \text{and} \quad G : \Sigma_b \to \Sigma_a$$

by

$$F((x_i)_{i \in Z^+}) = (f(x_i))_{i \in Z^+} \quad \text{and} \quad G((y_i)_{i \in Z^+}) = (g(y_i))_{i \in Z^+}.$$

It can be checked that F is a homeomorphism with $F \circ \hat{a} = \hat{b} \circ F$, $F(\xi_0) = \eta_0$ and indeed $G \circ F = \hat{a}^n$.

5.9. Proof of 4.5. By Theorem 4 (I) in [4], there exist $f : X \to Y$, $g : Y \to X$ uniquely determined by the properties that $f(x_0) = y_0$, $g(y_0) = x_0$, $f_* = \phi$, $g_* = \psi$, $f \circ a = b \circ f$, $g \circ b = a \circ g$. Similarly as $\psi \circ \phi \circ a_*^n = a_*^n \circ \psi \circ \phi$ there exists a unique map $h : X \to X$ with $h(x_0) = x_0$, $h_* = \psi \circ \phi$, $h \circ a^n = a^n \circ h$. Consequently $h = g \circ f = a^n$. By a similar reasoning we find that $f \circ g = b^n$. Now it is easy to check that f, g are covering projections. The proof is complete in view of 4.3.

5.10. LEMMA. *Let $b: Y \to Y$ be an expanding map, $y_0 \in Y$ a fixed point of b. For any $\langle \lambda \rangle \in \pi_1(Y, y_0)$, there exists a fixed point $y_0' \in Y$ of b such that*

$$\mathrm{Ad}[\langle \lambda \rangle] \circ b_{\#} : \pi_1(Y, y_0) \to \pi_1(Y, y_0)$$

is conjugate to

$$b_{\#} : \pi_1(Y, y_0') \to \pi_1(Y, y_0').$$

PROOF. Let $q: \bar{Y} \to Y$ be the universal covering projection onto Y. Let Γ be the group of covering transformations associated with $q: \bar{Y} \to Y$, that is the group of homeomorphisms $g: \bar{Y} \to \bar{Y}$ with the property that $q \circ g = q$. Given $\bar{y}_0 \in q^{-1}(y_0)$ there exists a natural isomorphism

$$\delta: \pi_1(Y, y_0) \to \Gamma$$

such that for any $\langle \lambda \rangle \in \pi_1(Y, y_0)$, δ is the unique covering transformation associated with $q: \bar{Y} \to Y$ sending $\bar{y}_0$ to $\bar{\lambda}(0)$ where $\bar{\lambda}$ is the unique path in $\bar{Y}$ with $q \circ \bar{\lambda} = \lambda$ and $\lambda(1) = \bar{y}_0$. If $\bar{b}: \bar{Y} \to \bar{Y}$ is the unique homeomorphism with $q \circ \bar{b} = b \circ q$ and $\bar{b}(\bar{y}_0) = \bar{y}_0$, it can be checked that

$$(*) \qquad \delta \circ b_{\#} = \beta \circ \delta$$

where $\beta: \Gamma \to \Gamma$ is defined by

$$\beta(g) = \bar{b} \circ g \circ (\bar{b})^{-1}$$

for any $g \in \Gamma$. As a consequence of $(*)$ we obtain

$$\delta \circ (\mathrm{Ad}[\langle \lambda \rangle] \circ b_{\#}) = (\mathrm{Ad}[\delta(\langle \lambda \rangle)] \circ \beta) \circ \delta.$$

We note that

$$\mathrm{Ad}[\delta(\langle \lambda \rangle)] \circ \beta(g) = \bar{b}' \circ g \circ (\bar{b}')^{-1}$$

where

$$\bar{b}' = \delta(\langle \lambda \rangle) \circ \bar{b}.$$

Lifting the riemannian metric on Y, with respect to which b is expanding, to $\bar{Y}$ so that $q: \bar{Y} \to Y$ is a local isometry and each $g \in \Gamma$ is an isometry of riemannian manifolds, we find that $\bar{b}$ and $\bar{b}'$ are expanding diffeomorphisms. Let $\bar{y}_0'$ is a fixed point of $\bar{b}'$. If $y_0' = q(\bar{y}_0')$, then y_0' is a fixed point of b and there is a natural isomorphism

$$\delta' : \pi_1(Y, y_0') \rightarrow \Gamma$$

such that

$$\delta' \circ b_* = \beta' \circ \delta'$$

where

$$\beta' : \Gamma \rightarrow \Gamma$$

is defined by

$$\begin{aligned}\beta'(g) &= \bar{b}' \circ g \circ (\bar{b}')^{-1} \\ &= \mathrm{Ad}[\delta(\langle \lambda \rangle)] \circ \beta(g).\end{aligned}$$

Consequently if we write

$$h = \delta^{-1} \circ \delta' : \pi_1(Y, y_0') \rightarrow \pi_1(Y, y_0)$$

we obtain

$$\begin{aligned}h \circ b_* &= \delta^{-1} \circ \delta' \circ b_* \\ &= \delta^{-1} \circ \beta' \circ \delta' \\ &= \delta^{-1} \circ \mathrm{Ad}[\delta(\langle \lambda \rangle)] \circ \beta \circ \delta' \\ &= \delta^{-1} \circ \delta \circ \mathrm{Ad}[\langle \lambda \rangle] \circ \delta^{-1} \circ \beta \circ \delta' \\ &= \mathrm{Ad}[\langle \lambda \rangle] \circ b_* \circ \delta^{-1} \circ \delta' \\ &= (\mathrm{Ad}[\langle \lambda \rangle] \circ b_*) \circ h.\end{aligned}$$

5.11. Proof of 4.6. Necessity is established in 4.2. Assume that $a_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$ is weakly shift equivalent to $b_* : \pi_1(Y, y_0) \rightarrow \pi_1(Y, y_0)$. There exists $\langle \lambda \rangle \in \pi_1(Y, y_0)$ such that $a_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$ is shift equivalent to $\mathrm{Ad}[\langle \lambda \rangle] \circ b_* : \pi_1(Y, y_0) \rightarrow \pi_1(Y, y_0)$. By 5.10 there exists a fixed point y_0' of b such that $\mathrm{Ad}[\langle \lambda \rangle] \circ b_* : \pi_1(Y, y_0) \rightarrow \pi_1(Y, y_0)$ is conjugate to $b_* : \pi_1(Y, y_0') \rightarrow \pi_1(Y, y_0')$. Consequently $a_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$ is shift equivalent to $b_* : \pi_1(Y, y_0') \rightarrow \pi_1(Y, y_0')$. By 4.5 σ_a is topologically conjugate to σ_b.

6. Expanding maps in the Klein bottle

Klein bottle is the compact quotient manifold $K = \mathbf{R}^2/G$ where G is the group of affine transformations of $\mathbf{R}^2$ generated by $A, B : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$A((x, y)) = (x + 1, y),$$

$$B((x, y)) = (-x, y + 1/2),$$

for any $(x, y) \in \mathbf{R}^2$. K has k_0, the image of $(0, 0) \in \mathbf{R}^2$ under the quotient map as a natural distinguished point. $\pi_1(K, k_0)$ can be identified with G which can be presented as the abstract group with generators A, B subject to the relation $AB = BA^{-1}$. Each element of G can be uniquely represented in the form A^pB^q, $p, q \in Z$. Let $\phi : G \to G$ be an endomorphism. There exist $p, p', q, r \in Z$ such that

$$\phi(A) = A^pB^{p'},$$

$$\phi(B) = A^qB^r.$$

As

$$\phi(AB) = \phi(BA^{-1})$$

we have

$$\phi(A)\phi(B) = \phi(B)\phi(A)^{-1},$$

hence

$$A^pB^{p'}A^qB^r = A^qB^rB^{-p'}A^{-p}$$

and

$$A^{p+(-1)^{p'}q}B^{p'+r} = A^{q-(-1)^{r-p'}p}B^{r-p'}.$$

Therefore

$$p' = 0,$$

$$p = -(-1)^rp.$$

Thus it is seen that each endomorphism ϕ of G is of the form

$$\phi(p,q,r):\begin{cases} A \to A^p \\ B \to A^q B^r \end{cases}$$

where $p, q, r \in Z$, r is odd unless $p = 0$. It can be checked that $\phi(p, q, r)$ is injective iff $p \neq 0$. Furthermore,

$$\phi(p,q,r)\circ\phi(p',q',r') = \phi(pp', q + q'p, rr'),$$

$$(\phi(p,q,r))^n = \phi(p^n, q(1 + p + \cdots + p^{n-1}), r^n).$$

For any $p, q, r \in Z$, $p \neq 0$, r odd consider

$$f(p,q,r): \mathbf{R}^2 \to \mathbf{R}^2$$

defined by

$$f(p,q,r)((x,y)) = (px + \theta(y), ry)$$

where $\theta: \mathbf{R} \to \mathbf{R}$ is any differentiable function with the properties

$$\theta(0) = 0,$$

$$\theta(y + 1/2) = -\theta(y) + q$$

for any $y \in \mathbf{R}$. (For instance, $\theta(y) = q \sin^2 \pi y$.) $f(p, q, r)$ is a diffeomorphism and satisfies

$$f(p,q,r)\circ A = A^p \circ f(p,q,r),$$

$$f(p,q,r)\circ B = A^q \circ B^r \circ f(p,q,r).$$

Hence $f(p, q, r)$ is the lifting of a covering projection

$$a(p,q,r): K \to K$$

with

$$a(p,q,r)(k_0) = k_0$$

and

$$(a(p,q,r))_* = \phi(p,q,r).$$

We notice that if $|p|, |r| \neq 0, 1$, then $a(p, q, r)$ is an expanding map. Conversely, if $\phi(p, q, r)$ is induced by an expanding map, then $|p|, |r| \neq 0, 1$:

If a is expanding and $a_* = \phi(1, q, r)$ for some $q, r \in Z$ then $A \in \bigcap_{i=0}^{\infty} a_*^i \pi_1(K, k_0)$ which is impossible by Proposition 4 in [4]. If a is expanding and $a_* = \phi(-1, q, r)$, then so is a^2 and

$$(a^2)_* = (a_*^2) = (\phi(-1, q, r))^2 = \phi(1, 0, r^2)$$

which was seen to be impossible. Hence $|p| \neq 1$. Similarly $|r| \neq 1$. If $a(p, q, r)$ is expanding then it is a covering projection. Thus $\phi(p, q, r)$ is injective and $p \neq 0$.

Finally if $a: K \to K$ is expanding it is topologically conjugate to some $a(p, q, r)$: Let $k \in K$ be a fixed point of a. If h is any diffeomorphism of K with $h(k_0) = k$ (which exists as K is a smooth manifold) then $a' = h^{-1} \circ a \circ h$ is an expanding map with k_0 as a fixed point. There exist $p, q, r \in Z^+$ such that $(a')_* = \phi(p, q, r)$. By Theorem 4 in [4] a' is topologically conjugate to $a(p, q, r)$.

Let $\Sigma(p, q, r) = \varprojlim (K, a(p, q, r))$ and $\sigma(p, q, r)$ be the shift on $\Sigma(p, q, r)$. By 4.6, the problem of classification of the shifts $\sigma(p, q, r)$ up to topological conjugacy is equivalent to the problem of classification of the group endomorphisms $\sigma(p, q, r)$ up to weak shift equivalence.

In the following $\mathrm{pr}(x)$ denotes the set of prime factors of $x \in Z$.

6.1. Proposition. *$\phi(p, q, r)$ is shift equivalent to $\phi(p', q', r')$ iff $p = p'$, $r = r'$ and $p - 1 \mid qL - q'$ for some $L \in Z$ with* $\mathrm{pr}(L) \subseteq \mathrm{pr}(p)$.

Proof. Assume that $\phi(p, q, r)$ is shift equivalent to $\phi(p', q', r')$. Let $\phi(L, M, N)$ and $\phi(L', M', N')$ effect the shift equivalence. As $\phi(L, M, N)$ has to be injective $L \neq 0$. On the other hand, $N \neq 0$ as N is odd. We have

$$\phi(L, M, N) \circ \phi(p, q, r) = \phi(p', q', r') \circ \phi(L, M, N),$$

that is,

$$(*) \qquad \phi(Lp, M + qL, Nr) = \phi(p'L, q' + Mp', Nr')$$

and

$$\phi(L', M', N') \circ \phi(L, M, N) = (\phi(p, q, r))^n,$$

that is,

$$(**) \qquad \phi(L'L, M' + ML', N'N) = \phi(p^n, q(1 + p + \cdots + p^{n-1}), r^n)$$

for some $n \in Z^+$. From (*) we obtain

$$Lp = p'L,$$

$$Nr = r'N,$$

$$M + qL = q' + Mp';$$

from (**)

$$L \mid p^n.$$

Consequently $p = p'$, $r = r'$, $p - 1 = p' - 1 \mid qL - q'$ for some $L \in Z$ with $\mathrm{pr}(L) \subseteq \mathrm{pr}(p)$. Conversely, if the above relations are valid choose sufficiently large $n \in Z^+$ such that $L \mid p^n$. Write $L' = p^n/L$. There exists $M \in Z$ such that

$$qL - q' = M(p - 1) = M(p' - 1)$$

hence

(1) $$M + qL = q' + Mp'.$$

As

$$p - 1 \mid qL - q' = qp^n/L' - q'$$

we have

$$p - 1 \mid qp^n - q'L'$$

which implies

$$p - 1 \mid q'L' - q$$

as $p - 1 \mid p^n - 1$. There exists $M' \in Z$ such that

$$M'(p - 1) = q'L' - q$$

hence

(2) $$M' + q'L' = q + M'p.$$

From (1) and (2) and the fact that $LL' = p^n$, $p = p'$, $r = r'$ we obtain

(3) $$M + M'L = q'(1 + p' + \cdots + p'^{n-1}),$$

(4) $$M' + ML' = q(1 + p + \cdots + p^{n-1}).$$

Let N, N' be any integers with

(5) $$N \cdot N' = r^n = r'^n,$$

then

$$\phi(L, M, N) \circ \phi(p, q, r) = \phi(p', q', r') \circ \phi(L, M, N),$$

$$\phi(L', M', N') \circ \phi(p', q', r') = \phi(p, q, r) \circ \phi(L', M', N'),$$

$$\phi(L', M', N') \circ \phi(L, M, N) = (\phi(p, q, r))^n,$$

$$\phi(L, M, N) \circ \phi(L', M', N') = (\phi(p', q', r'))^n,$$

by (1), (2), (3), (4), (5). Therefore $\phi(L, M, N)$, $\phi(L', M', N')$ effect a shift equivalence of $\phi(p, q, r)$ to $\phi(p', q', r')$.

6.2. Proposition. *$\phi(p, q, r)$ is weakly shift equivalent to $\phi(p', q', r')$ iff*
(a) $|p| = |p'|$, $r = r'$,
(b) *p is even or q and q' have the same parity.*

Proof. We notice that

$$\mathrm{Ad}[A^u B^v] \circ \phi(p, q, r) = \phi((-1)^v p, (-1)^v q + 2u, r)$$

for any $p, q, r \in Z$. Hence $\phi(p, q, r)$ is weakly shift equivalent to $\phi(p', q', r')$ iff there exist $u, v, L \in Z$ such that

$$p = (-1)^v p', \quad r = r', \quad \mathrm{pr}(L) \subseteq \mathrm{pr}(p)$$

and

$$p - 1 \mid qL - (-1)^v q' - 2u.$$

Equivalently, $\phi(p, q, r)$ is weakly shift equivalent to $\phi(p', q', r')$ iff $|p| = |p'|$, $r = r'$ and there exist $u, L \in Z$ such that

$$\mathrm{pr}(L) \subseteq \mathrm{pr}(p)$$

and

(∗) $$p - 1 \mid qL - q' - 2u.$$

If p is even (∗) is satisfied by taking $L = 2$ and

$$2(q-u)-q'=p-1 \qquad \text{if } q' \text{ is odd},$$

$$2(q-u)-q'=2(p-1) \quad \text{if } q' \text{ is even}.$$

If p is odd then so is L and $(*)$ can be satisfied iff q and q' have the same parity.

6.3. COROLLARY. *Let* $|p|,|p'| \neq 1, 0$, $|r|,|r'| \neq 1$. $\sigma(p, q, r)$ *is topologically conjugate to* $\sigma(p', q', r')$ *iff*

(a) $|p| = |p'|$, $r = r'$,

(b) *p is even or q and q' have the same parity.*

PROOF. Immediate consequence of 4.6 and 6.2.

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